

# Anomalies and fermion content of grand unified theories in extra dimensions

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The restrictions imposed by anomaly cancellation on the chiral fermion content of nonsupersymmetric gauge theories based on various groups are studied in spacetime dimension  $D=6, 8$ , and  $10$ . In particular, we show that the only mathematically consistent chiral  $SU(5)$  theory in  $D=6$  contains three nonidentical generations.

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## I. INTRODUCTION

Despite its numerous successes, the standard model of particle physics is far from being satisfactory. The fermion sector is particularly puzzling. Among other problems, one may wonder why there are so many different fermions, with apparently arbitrary quantum numbers under  $SU(3) \otimes SU(2) \otimes U(1)$ , and why it is possible to divide them into three generations.

The first question can be partially answered: the quantum numbers ensure the cancellation of all potentially dangerous chiral anomalies [1]. Historically, the latter were discovered [2] while most fermions of the standard model were already known experimentally: the absence of anomalies was more a way of checking the consistency *a posteriori* than a predictive tool. Nevertheless, the anomaly cancellation condition led to an alternative prediction of the existence of the  $c$  quark [3]. Furthermore, it has been shown that, with some additional assumption, namely, that the fermions may only be  $SU(2)[SU(3)]$  singlets or doublets [triplets], an anomaly-free fermion content with the minimal number of fields fits precisely within one generation of the standard model [4]. However, four-dimensional anomalies do not explain why there should be three generations in nature.

Various explanations for the existence of several generations have been proposed. In theories with extra dimensions, for instance, the number of generations can be related, through the index theorem, to the topology of the compact manifold [5] or to the winding of some field configuration (see e.g., [6]). In the Connes-Lott version of the standard model in noncommutative geometry, the existence of spontaneous chiral symmetry breaking requires the existence of more than one generation [7]. Recently, it has been proposed that anomalies could actually yield a constraint on the number of generations, provided the cancellation of anomalies takes place in an  $SU(3) \otimes SU(2) \otimes U(1)$  theory that lives in six spacetime dimensions (6D) [8].

In this paper, we shall further investigate the anomaly cancellation condition in arbitrary spacetime dimension, extending the discussion of [8] to larger groups containing  $SU(3) \otimes SU(2) \otimes U(1)$ . We shall only consider the case of even dimensions: in odd dimensions, there is no chirality, hence no chiral anomaly, and the closest equivalent, the parity anomaly, can be canceled by a Chern-Simons term in the action [9]. Also, since anomalies yield no information on vectorlike generations, we shall only derive constraints on the number  $n_g$  of chiral generations.

In order to make our paper self-contained, we first review in Sec. II the different types of chiral anomalies which will be relevant in the sequel. In Sec. III, we impose the absence of anomaly in (nonsupersymmetric) gauge theories based on any of the groups  $SU(3) \otimes SU(2) \otimes U(1)$ ,  $SU(5)$ ,  $SO(10)$ , and  $E_6$ , in dimensions  $D=6, 8$ , and  $10$ , and deduce in each case the possible fermion contents. Among the various cases, the  $SU(5)$ -based theories are the most constrained: in particular, in six dimensions, only theories with  $n_g=0 \bmod 3$  generations are anomaly-free. Let us emphasize right away that, because charge conjugation does not change chirality in  $D=6$ , this  $SU(5)$  solution is not a trivial generalization of the well-known 4D construction. In Sec. IV, we study the possible embedding of the 4D standard model in this 6D  $SU(5)$  theory. We give some conclusions and prospects for future works in Sec. V. Finally, some useful results and demonstrations are given in the Appendixes A and B.

## II. SHORT REVIEW OF CHIRAL ANOMALIES

A symmetry is said to be anomalous if it exists at the classical level, but does not survive quantization. In some cases anomalies are welcome, as in the  $\pi^0$  decay [2]. These harmless anomalies are always associated with global symmetries of the Lagrangian. In opposition, anomalies which affect local symmetries, in particular gauge symmetries, jeopardize the theory consistency. Such anomalies spoil renormalizability; but even in the case of effective, *a priori* non-renormalizable theories, they destroy unitarity, leading to theories without predictive power [10]. Consistent models should therefore either contain none of these anomalies, or automatically cancel them [11,12]. Conversely, the cancellation condition gives useful constraints on the structure of a theory, and especially on its fermion content [13], as we shall recall in Sec. III.

We shall be interested in the so-called chiral anomalies, which involve chiral fermions in the presence of gauge fields and/or gravitons. They can be divided in two classes: local (Sec. II A) and global (or nonperturbative; see Sec. II B), according to whether they can be calculated perturbatively or not.

### A. Local anomalies

Local anomalies are related to infinitesimal gauge and/or coordinate transformations. They arise from a typical kind of Feynman diagram, which leads to a possible non-

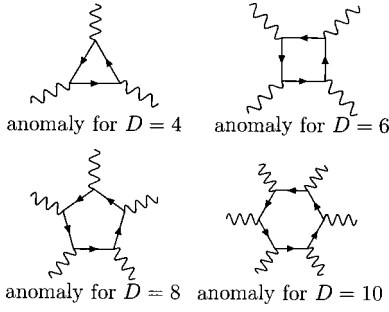


FIG. 1. Anomalous diagrams in  $D=4, 6, 8$ , and  $10$  dimensions. Each external leg stands for any of the gauge bosons of the theory, while the fermions circulating in the internal lines can be in any relevant representation of the gauge group.

conservation of the gauge symmetry current or the energy-momentum tensor. The topology of these diagrams depends on the spacetime dimension  $D$ . In  $D=4$ , these are the well-known triangle diagrams [2]. In 6-, 8-, and 10-dimensional theories, the corresponding possibly anomalous diagrams are respectively the so-called box, pentagonal, and hexagonal diagrams [14,15], represented together with the triangle diagram in Fig. 1.

One can distinguish three types of local anomalies, according to the nature of the external legs of the anomalous diagrams. Diagrams with only gauge bosons correspond to the pure gauge anomaly. On the other hand, when all external legs are gravitons, the diagram yields the pure gravitational anomaly [16]. Finally, the mixed anomaly correspond to diagrams with both gauge bosons and gravitons [16,17]. These various types are illustrated, in the case of  $D=6$  dimensions, in Fig. 2.

### 1. Pure gauge anomaly

In the pure gauge case, the anomaly is proportional to a group factor, which multiplies a Feynman integral:

$$\sum_{L_D} \text{STr}(T^a T^b \dots T^{D/2+1}) - \sum_{R_D} \text{STr}(T^a T^b \dots T^{D/2+1}), \quad (2.1)$$

where the notation  $\text{STr}$  means that the trace is performed over the symmetrized product of the gauge group generators  $T^a$ . This symmetrization is related to the Bose-Einstein statistics of the interaction fields. The sums run over all left- and right-handed (in the  $D$ -dimensional sense; see Appendix A) fermions of the theory belonging to the representation  $T^a$ .

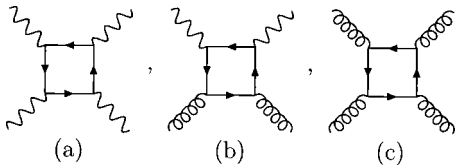


FIG. 2. Local [(a) pure gauge, (b) mixed, and (c) pure gravitational] anomalies in 6 dimensions. Wavy external legs stand for gauge bosons, while loopy legs stand for gravitons.

The symmetrized traces of Eq. (2.1) can be expressed in terms of traces over products of the generators  $t^a$  of the fundamental representation, and can sometimes be factorized. The first property reduces the number of traces which must be calculated, provided the coefficients relating the traces over arbitrary generators to the traces over the  $t^a$  are known [14,18]. The second property is due to the existence of basic (i.e., non-factorizable) traces, the number of which, and the number of generators they involve, being related to the rank and the Casimir operators of the group. A simple example is  $SU(2)$ , which is of rank 1, with the unique Casimir operator  $(T)^2$ ; a trace involving more than two generators can be factorized:

$$\text{STr}(T^a T^b T^c) \propto S(\text{Tr}(T^a T^b) \text{Tr}(T^c)) = 0, \quad (2.2)$$

$$\text{STr}(T^a T^b T^c T^d) \propto S(\text{Tr}(T^a T^b) \text{Tr}(T^c T^d)). \quad (2.3)$$

In other words, the triangle diagram for  $[SU(2)]^3$  (sometimes called cubic anomaly) vanishes for any  $SU(2)$  representation of fermions. Equation (2.3) states that the quartic  $SU(2)$  anomaly is factorizable.

The pure gauge anomaly (2.1) vanishes either if the group is “safe” [11,12], as is the case for  $SU(2)$  in four dimensions, or if the fermion content of the theory is properly chosen. Nevertheless, it is possible that part of the anomaly is zero thanks to the matter content, while the remaining part can be canceled by an additional tensor, through the Green-Schwarz mechanism [19], as will be discussed later.

### 2. Pure gravitational anomaly

The gravitational anomaly [16] represents a breakdown of general covariance, or, equivalently, of the conservation of the energy-momentum tensor, due to parity-violating couplings between fermions and gravitons. In particular, chiral fermions obviously violate parity, and lead to such anomalies. A necessary and sufficient condition for the absence of local gravitational anomaly is therefore the identity of the numbers of left- and right-handed fermions:

$$N_{L_D} - N_{R_D} = 0. \quad (2.4)$$

As we recall in Appendix A 2, in dimension  $D=4k$  charge conjugation flips chirality, while it does not in  $D=4k+2$ . Therefore, a left-handed Weyl fermion field contains a left-handed particle and an antiparticle with opposite (identical) chirality in  $D=4k$  ( $D=4k+2$ ). Such a field is, from the gravitational point of view, vector-like in dimension  $4k$ , while it is chiral in dimension  $4k+2$ . Thus, the local gravitational anomaly always vanishes if the spacetime dimension is  $D=4k$ .

Note that a gravitational anomaly can always be canceled by the addition of the right number of gauge singlet chiral fields. This addition obviously does not affect the gauge and mixed anomalies. Therefore, this anomaly does not yield a very stringent constraint from the phenomenological point of view.

### 3. Mixed anomaly; Green-Schwarz mechanism

The mixed gauge-gravitational anomaly [Fig. 2, diagram (b)] is proportional to the product of a gauge group factor and a gravitational term. The latter vanishes when the number of gravitons is odd [16].

When the mixed anomaly does not vanish thanks to group properties or an appropriate fermion choice, it may still be canceled through the Green-Schwarz mechanism [19]. This mechanism relies on the existence, in dimension  $D \geq 6$ , of tensors which, with properly chosen couplings, can cancel anomalies proportional to the trace of the product of  $k$  generators, with  $2 \leq k \leq D/2 - 1$ . The anomalies which can be canceled in this way, either mixed or pure gauge, are called reducible, and the others, irreducible. Since gauge and mixed anomalies can be factorized, the factorization may amount to dividing the anomaly into a reducible part, which can be canceled through the Green-Schwarz mechanism, and an irreducible part,<sup>1</sup> which necessitates some appropriate fermion content.

### B. Global anomalies

In addition to the local anomalies discussed previously, there are also nonperturbative anomalies, which cannot be obtained from a perturbative expansion, and will be called global in the following, although they are related to local symmetries. Two types of such anomalies can arise, related either to gauge invariance or to gravity.

The global gauge anomaly [21] occurs when there exist gauge transformations which cannot be deduced continuously from the identity, in the presence of chiral fermions. In other terms, the anomaly arises when the  $D$ -th homotopy group of the gauge group  $G$ ,  $\Pi_D(G)$ , is nontrivial. The anomaly then leads to mathematically inconsistent theories in which all physical observables are ill-defined.

This anomaly vanishes only if the matter content of the theory is appropriate. More precisely, if  $\Pi_D(G) \neq 0$ , the cancellation of the anomaly constrains the numbers  $N(p_{L_D})$  and  $N(p_{R_D})$  of left- and right-handed  $p$ -uplets:

$$\Pi_D(G) = \mathbb{Z}_{n_D} \Rightarrow c_D [N(p_{L_D}) - N(p_{R_D})] = 0 \mod n_D, \quad (2.5)$$

where  $c_D$  is an integer whose value depends on the space-time dimension  $D$ , the gauge group  $G$ , and the representation of  $G$  the fermions belong to.

In the case of the  $SU(2)$  global anomaly in  $D=4$  dimensions,  $\Pi_4(SU(2)) = \mathbb{Z}_2$ , so the anomaly cancellation condition reads, for the fundamental representation ( $c_4 = 1$ ),  $N(2_{L_4}) - N(2_{R_4}) = 0 \mod 2$ , where  $N(2_{L_4})$  and  $N(2_{R_4})$  are the numbers of left- and right-handed Weyl fermions which are doublets under  $SU(2)$  [21].

<sup>1</sup>In fact, a *sufficient* condition for the existence of an irreducible anomaly for the gauge group  $G$  is  $\Pi_{D+1}(G) = \mathbb{Z}$ , where  $\Pi_{D+1}(G)$  is the  $(D+1)$ -th homotopy group of  $G$  [20].

Coordinate transformations which cannot be reached continuously from the identity give rise to possible global gravitational anomalies [16]. In a  $(4k+2)$ -dimensional space-time, these anomalies vanish when condition (2.4) holds: the cancellation of the local gravitational anomaly automatically ensures that the global one is zero. In  $D=8k$  dimensions, the anomaly vanishes only if the number of (spin  $\frac{1}{2}$ ) Weyl fermions coupled to gravity is even; otherwise, the theory is inconsistent. Note that in that case, there is no local gravitational anomaly. This is similar to the possibility of global gauge anomalies for  $SU(2)$  in 4 dimensions, while there is no corresponding local anomaly.

Finally, there is another important feature of anomalies, which we shall encounter in the following, related to symmetry breaking. When a symmetry is spontaneously broken, from a larger group  $G$  into a subgroup  $H$ , anomalies may neither be created nor destroyed, and propagate from  $G$  to  $H$ . However, the type of the anomaly may change: a local anomaly in  $G$  can become a global anomaly of  $H$ . For instance, the  $SU(2)$  global anomaly in  $D=4$  discussed above corresponds to a local  $SU(3)$  anomaly [22,23].

### III. CONSTRAINTS FROM ANOMALIES

In this section, we use chiral anomalies to restrict the fermion content of (nonsupersymmetric) theories based on various gauge groups in different spacetime dimensions. More precisely, we shall limit ourself to the study of the possible anomalies in every group of the familiar symmetry breaking sequence<sup>2</sup>

$$E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1). \quad (3.1a)$$

For each group, we shall focus on the lowest dimensional representations which might be relevant for the standard model content, namely the 27 of  $E_6$ , the 16 of  $SO(10)$ , the  $\bar{5}$  and 10 of  $SU(5)$ , and the usual doublets and triplets of  $SU(3) \otimes SU(2) \otimes U(1)$ . These are related by

$$27 \rightarrow 1 \oplus 10 \oplus 16 \rightarrow 1 \oplus (5 \oplus \bar{5}) \oplus (1 \oplus \bar{5} \oplus 10),$$

$$\bar{5} \oplus 10 \rightarrow (\mathcal{D}, \mathcal{L}) \oplus (\mathcal{Q}, \mathcal{U}, \mathcal{E}), \quad (3.1b)$$

where the second line are respectively one generation of  $SU(5)$  and one of the standard model. The quantum numbers of the  $SU(3) \otimes SU(2) \otimes U(1)$  fermions  $(\mathcal{Q}, \mathcal{U}, \mathcal{D}, \mathcal{L}, \mathcal{E})$  are

$$\left(3, 2, \frac{1}{3}\right), \quad \left(\bar{3}, 1, \frac{-4}{3}\right), \quad \left(\bar{3}, 1, \frac{2}{3}\right), \quad (1, 2, -1), \quad (1, 1, 2). \quad (3.1c)$$

<sup>2</sup>In particular, we do not consider the more string-inspired gauge groups  $SO(32)$  or  $E_8 \otimes E_8$  in  $D=10$  or their compactifications to lower dimensions [19,24].

In  $D=4k$  dimensions, one may equivalently replace  $(\bar{3}, 1, -4/3)$  and  $(\bar{3}, 1, \frac{2}{3})$  by  $(3, 1, \frac{4}{3})$  and  $(3, 1, -2/3)$ , i.e., replace the left-handed fields  $\mathcal{U}_{L_D}$  and  $\mathcal{D}_{L_D}$  with their right-handed charge conjugates  $(\mathcal{U}^c)_{R_D}$ ,  $(\mathcal{D}^c)_{R_D}$ . However, since the  $(4k+2)$ -dimensional charge conjugation does not flip chirality, the choices are no longer equivalent in  $D=6$  or  $10$ .

Similar studies have been carried out before, under more restrictive assumptions, either without the benefit of the Green-Schwarz mechanism to cancel reducible anomalies [25], or using only one representation per group to cancel the reducible anomalies [26,27].

### A. Constraints in $D=6$ dimensions

As is well known, all groups we mentioned above admit anomaly-free fermion content in four dimensions. In every case, the anomalies cancel within a generation, and thus they do not restrict the number of generations. We shall show that the situation is rather different in  $D=6$ , where anomalies yield stronger constraints than in  $D=4$ . We do not impose any *a priori* condition on the (six-dimensional) chiralities of the representations, which are not constrained by experimental results.

#### 1. $SU(3) \otimes SU(2) \otimes U(1)$ anomalies

In the case of the group  $SU(3) \otimes SU(2) \otimes U(1)$ , the anomalies which may arise in six dimensions are

local gauge anomalies, the only irreducible one being  $[SU(3)]^3 U(1)$ , and possibly  $[U(1)]^4$  if  $\text{Tr } Y^2$  vanishes, where  $Y$  is the generator of  $U(1)$ . If  $\text{Tr } Y^2 \neq 0$ ,  $[U(1)]^4$  is reducible, as are  $[SU(3)]^4$ ,  $[SU(2)]^4$ ,  $[SU(3)]^2 [SU(2)]^2$ , and  $[SU(2)]^2 [U(1)]^2$ , and they all can be canceled by at most three Green-Schwarz tensors;

mixed anomalies, represented in Fig. 2(b), where the gauge bosons belong to  $SU(3)$ ,  $SU(2)$ , or  $U(1)$ , are reducible, and canceled by the same tensors which are used for the pure gauge anomalies.

local gravitational anomalies;

global gauge anomalies, since  $\Pi_6(SU(3)) = \mathbb{Z}_6$  and  $\Pi_6(SU(2)) = \mathbb{Z}_{12}$ .

All in all, there are four conditions which must be satisfied: the sums of the hypercharges over the  $SU(3)$  triplets and antitriplets must vanish. Then, there must be as many left- as right-handed fields. Finally, using Eq. (2.5) with  $c_6 = 2$  for the  $SU(2)$  doublets and  $c_6 = 1$  for the  $SU(3)$  triplets [28], we find that the numbers of doublets and triplets must satisfy

$$N(2_{L_6}) - N(2_{R_6}) = 0 \pmod{6}, \quad (3.2a)$$

$$N(3_{L_6}) - N(3_{R_6}) = 0 \pmod{6}. \quad (3.2b)$$

As mentioned above, the extension of the standard model in  $D=6$  dimensions gives several inequivalent models with different assignments of the quantum numbers. The consistency of the theory, with a given assignment, has been studied previously, under the assumption that local anomalies cancel within a single generation [8]. This leads to specific

chirality choices for the six-dimensional  $SU(3) \otimes SU(2) \otimes U(1)$  fermions, and to the introduction of an additional singlet in each generation. An important result is that it is necessary to have more than one generation, in order to cancel the global anomalies [8]. Furthermore, if the generations are identical, i.e., have the same chiralities, their number  $n_g$  is a multiple of 3, up to an arbitrary number of vector-like pairs of generations. However, relaxing the requirement that local anomalies cancel in each generation, there arise other anomaly-free solutions with three generations.

It is also possible to keep our original quantum number assignment, Eq. (3.1c). With that choice, there is no anomaly-free solution with only one or two generations:  $n_g \geq 3$ . There are several solutions with the “minimal” three-generation content. For instance, one might take three copies of the locally anomaly-free generation composed of  $\mathcal{Q}_{L_6}, \mathcal{U}_{R_6}, \mathcal{D}_{R_6}, \mathcal{L}_{L_6}, \mathcal{E}_{R_6}$ , and a right-handed singlet (for the local gravitational anomaly). A drawback of this solution is that it cannot be embedded in a larger gauge group since  $\mathcal{D}$  and  $\mathcal{L}$ , for instance, have opposite chiralities.

Another possible solution, with three nonidentical generations, is

$$\begin{aligned} &(\mathcal{Q}_{L_6}, \mathcal{U}_{L_6}, \mathcal{D}_{L_6}, \mathcal{L}_{L_6}, \mathcal{E}_{L_6}), \\ &(\mathcal{Q}_{L_6}, \mathcal{U}_{L_6}, \mathcal{D}_{L_6}, \mathcal{L}_{L_6}, \mathcal{E}_{L_6}), \\ &(\mathcal{Q}_{R_6}, \mathcal{U}_{R_6}, \mathcal{D}_{L_6}, \mathcal{L}_{L_6}, \mathcal{E}_{R_6}), \end{aligned} \quad (3.3)$$

plus  $SU(3) \otimes SU(2) \otimes U(1)$  singlets. In that case, local anomalies do not vanish within a single generation, but rather between one generation and two copies of  $\mathcal{D}_{L_6}$  and  $\mathcal{L}_{L_6}$ . To obtain three full generations, we used the freedom to add vector-like, anomaly-free representations, i.e.,  $\mathcal{Q}$ ,  $\mathcal{U}$ , and  $\mathcal{E}$ . An obvious problem is that the latter could be given Dirac mass terms and be decoupled from the low-energy theory. A simple, however admittedly inelegant, way to prevent this is to assign some discrete symmetry to these extra fields, such as  $\mathcal{Q}_{R_6} \rightarrow -\mathcal{Q}_{R_6}$  while  $\mathcal{Q}_{L_6} \rightarrow \mathcal{Q}_{L_6}$ ; it is anyway necessary to implement such a symmetry in order to recover a 4D chiral theory by compactification on an orbifold.

Finally, note that there are still other 3-generation solutions, as well as solutions with for instance  $n_g = 5$ , which are not replications of solutions with  $n_g = 3$ . Nevertheless, we wish to emphasize that there is a feature which does not depend on the quantum number assignment. If one requires identical generations, then the only anomaly-free solutions consist of  $n_g = 0 \pmod{3}$  generations, each of which must have no local anomaly: each generation only brings 2 or 4 (left- minus right-handed) doublets or triplets, while multiples of 6 are necessary, see Eqs. (3.2).

When this additional condition is not imposed, the number of generations is not strictly fixed by the anomaly cancellation requirement. This compels us to examine whether even more stringent conditions might be derived from larger groups in the sequence Eq. (3.1a).



TABLE I. Coefficients in the symmetrized trace factorization Eq. (3.5) for the lowest dimension  $SU(5)$  representations.

	$k$	$A_4(5,k)$	$A_4^{22}(5,k)$
5	1	1	0
10	2	-3	3
$\overline{10}$	3	-3	3
$\overline{5}$	4	1	0

## 2. $SU(5)$ anomalies in six dimensions

It was soon recognized that it might be possible to explain some of the arbitrary features of the standard model by embedding  $SU(3) \otimes SU(2) \otimes U(1)$  in a larger gauge group [29]. The minimal solution relying on a simple Lie group is the Georgi-Glashow  $SU(5)$  model [30], in which the standard model fermions are represented by  $(\overline{5} \oplus 10)$  generations.

Let us review the various possible anomalies of a 6D  $SU(5)$  theory, and, first of all, the local gauge anomaly. Since  $\Pi_7(SU(5)) = \mathbb{Z}$ , a single representation has an irreducible anomaly. Taking  $D=6$  in Eq. (2.1), the cancellation condition reads

$$\sum_{L_6} \text{STr}(T^a T^b T^c T^d) - \sum_{R_6} \text{STr}(T^a T^b T^c T^d) = 0. \quad (3.4)$$

These traces must be calculated for both representations  $\overline{5}$  and 10, relating them to the traces over the generators  $t^a$  of the fundamental  $SU(5)$  representation (we follow the notations of [14]):

$$\begin{aligned} \text{STr}(T_{(k)}^a T_{(k)}^b T_{(k)}^c T_{(k)}^d) &= A_4(5,k) \text{STr}(t^a t^b t^c t^d) \\ &+ A_4^{22}(5,k) S(\text{Tr}(t^a t^b) \text{Tr}(t^c t^d)), \end{aligned} \quad (3.5)$$

where  $k$  labels the representation under study:  $k=1$  for the fundamental representation 5,  $k=2$  for the 10,  $k=3$  for the  $\overline{10}$ , and  $k=4$  for the  $\overline{5}$ . The coefficients are given in Table I. Note that the coefficients in the case  $k=1$  are trivial.

Both traces in the right-hand side of Eq. (3.5) are nonvanishing, but only the trace over four matrices is irreducible. To cancel this first part, one must choose the fermion content of the theory appropriately. The simplest solution (apart from vector-like generations) made of  $\overline{5} \oplus 10$  representations consists in taking three  $(\overline{5})_{L_6}$ , two  $10_{L_6}$ , and a  $10_{R_6}$ : the resulting anomaly is proportional to

$$3A_4(5,4) + 2A_4(5,2) - A_4(5,2) = 3 + (-6) - (-3) = 0, \quad (3.6)$$

where the minus sign is due to the opposite chirality of the  $10_{R_6}$ . If we consider that a  $\overline{5}$  and a 10 form a generation,<sup>3</sup> this means that we need three *nonidentical* generations to

cancel the irreducible part of the gauge anomaly. Of course, one may add other identical copies of this set of three families: the number of generations is  $n_g = 0 \pmod{3}$ .

As stated in Sec. II A, the reducible part of the anomaly can be canceled through the Green-Schwarz mechanism, by introducing a self-dual, antisymmetric tensor [19,31]. The same tensor allows us to also cancel the mixed anomaly. This latter is proportional to  $\text{Tr}(T^a T^b)$ , see diagram (b) of Fig. 2, and does not vanish with our fermion choice.

Let us now consider the local gravitational anomaly. As recalled above, it vanishes provided the numbers of left- and right-handed fields are the same. The fermion content imposed by the cancellation of the gauge anomaly consists of  $5+5+5+10+10$  left- and 10 right-handed Weyl fermions. In addition, the self-dual antisymmetric tensor contributes for 28 Weyl right-handed fermions [16]. All in all, three additional left-handed fermions, necessarily singlets under  $SU(5)$ , are required to cancel the anomaly.

While there can be local anomalies of every type—gauge, mixed, and gravitational—unless the fermion content of the theory is carefully chosen, the theory cannot be spoiled by the global gauge anomaly, because the sixth homotopy group of  $SU(5)$  is trivial.

In conclusion, imposing the absence of anomaly for a chiral  $SU(5)$  theory in 6 dimensions fixes its gauged fermion content:

$$\left( \begin{array}{c} (\overline{5})_{L_6} \\ 10_{L_6} \end{array} \right) \left( \begin{array}{c} (\overline{5})_{L_6} \\ 10_{L_6} \end{array} \right) \left( \begin{array}{c} (\overline{5})_{L_6} \\ 10_{R_6} \end{array} \right), \quad (3.7)$$

and requires the introduction of a self-dual antisymmetric tensor and three left-handed singlets. This solution is not the *minimal* anomaly-free solution with  $\overline{5}$ s and 10s: we have added a vector-like 10 to obtain three full generations, and the remarks following Eq. (3.3) also apply here.

What does this solution we propose become when  $SU(5)$  is broken into  $SU(3) \otimes SU(2) \otimes U(1)$ ? Given the chirality assignments of Eq. (3.7), we actually recover the three chiral  $SU(3) \otimes SU(2) \otimes U(1)$  generations of Eq. (3.3). We shall explicitly check that this is indeed an anomaly-free set of three generations, although we already know it must be the case since no anomaly can have been created when  $SU(5)$  was broken.

As we have seen above, the only irreducible gauge anomaly is  $[SU(3)]^3 U(1)$ , since one easily checks that  $\text{Tr} Y^2 \neq 0$ . This anomaly is proportional to

$$\begin{aligned} 2 \times \left[ \frac{1}{3} + \frac{1}{3} - \left( \frac{-4}{3} + \frac{2}{3} \right) \right] &+ \left[ -\left( \frac{1}{3} + \frac{1}{3} \right) + (-1)^2 \left( \frac{-4}{3} - \frac{2}{3} \right) \right] \\ &= 0, \end{aligned} \quad (3.8)$$

where the factor  $(-1)^2$  reflects both right-handed chirality and  $A_3(3,2) = -1$  for the  $\overline{3}$ . The other, reducible anomalies are destroyed through the Green-Schwarz mechanism: the single tensor which was used to cancel the  $SU(5)$  anomalies somewhat splits into different parts, which in turn cancel the  $SU(3) \otimes SU(2) \otimes U(1)$  anomalies after breaking.

<sup>3</sup>We shall comment on this issue in Sec. V.

TABLE II. Possible  $SU(5)$  and  $SU(3) \otimes SU(2) \otimes U(1)$  anomalies in 6 dimensions.

$D=6$	$SU(5)$	$SU(3) \otimes SU(2) \otimes U(1)$
pure gauge	yes	yes
mixed	yes	yes
gravitational	yes	yes
global	no	yes

The  $SU(5)$  singlet fermions, which were introduced to cancel the gravitational anomaly, are now  $SU(3) \otimes SU(2) \otimes U(1)$  singlets. Since gravity is insensitive to the breaking of other interactions, the gravitational anomaly cancellation remains valid.

Global gauge anomalies, on the other hand, depend on the gauge group: while there is none in  $SU(5)$ , both  $SU(2)$  and  $SU(3)$  can possibly have such anomalies. As seen above, their cancellation requires 0 mod 6 doublets (left- minus right-handed) and 0 mod 6 triplets [see Eq. (2.5)]. Our  $SU(5)$ -inspired solution satisfies both conditions:  $N(2_{L_6}) - N(2_{R_6}) = 9 - 3 = 6$  and  $N(3_{L_6}) - N(3_{R_6}) = 9 - 3 = 6$ . From the point of view of  $SU(3) \otimes SU(2) \otimes U(1)$ , this is the condition which suggests some restriction on the number of generations. In  $SU(5)$ , the condition comes from the irreducible part of the gauge anomaly. This is a striking example of the change of nature of an anomaly when a group is broken.

Therefore, the anomaly-free three-generation  $SU(5)$  theory becomes an anomaly-free  $SU(3) \otimes SU(2) \otimes U(1)$  theory when the symmetry is broken, as expected. On the other hand, all other anomaly-free, six-dimensional  $SU(3) \otimes SU(2) \otimes U(1)$  theories do not originate from a  $SU(5)$  theory. This is for instance the case of the solution with 3 identical generations we mentioned in the discussion on  $SU(3) \otimes SU(2) \otimes U(1)$  anomalies, since the  $\mathcal{D}$  and  $\mathcal{L}$  have opposite chiralities, and cannot come from a single  $\bar{5}$ .

We have summarized in Table II the different anomalies which can affect  $SU(5)$  and  $SU(3) \otimes SU(2) \otimes U(1)$  theories in six dimensions. Note that while there is no global anomaly in  $SU(5)$ , there is one in the standard model group, which automatically vanishes for a  $SU(3) \otimes SU(2) \otimes U(1)$  matter content which comes from  $SU(5)$ .

### 3. Six-dimensional $SO(10)$ and $E_6$ anomalies

As we have just shown, a six-dimensional  $SU(5)$  theory is anomaly-free only if the number  $n_g$  of chiral generations is a multiple of 3, with specific chirality assignments for the various  $\bar{5}$  and 10 which yield the matter content of the standard model, plus additional  $SU(5)$  singlets. For the economical solution  $n_g=3$ , there are three such singlets.

A similar result was obtained in [8], where each chiral generation must be added a  $SU(3) \otimes SU(2) \otimes U(1)$  singlet to cancel the gravitational anomaly. In both cases, it is necessary to introduce as many gauge singlets as there are chiral generations, although it should be noted that the underlying reasons are different. In [8], the number of extra fermions is necessarily equal to the number of chiral families, since the

local anomalies are required to vanish within each generation. On the other hand, in the present paper, we do not impose this condition, and we still have to add 3  $SU(5)$  singlets to our 3 chiral generations.

This coincidence naturally leads to the idea that both models could be embedded in more fundamental theories based on a larger symmetry group. The first obvious candidate, which unifies the 15 Weyl fermions of a given generation of the standard model with a  $SU(5)$  or  $SU(3) \otimes SU(2) \otimes U(1)$  singlet in a single representation, is the orthogonal group  $SO(10)$ , with its spinorial 16 representation [32]. Next, we shall consider the case of the exceptional group  $E_6$ .

The case of  $SO(10)$  is rather different from the previous groups we considered. Since  $\Pi_7(SO(10))=Z$ , there must be an irreducible anomaly. Indeed,  $\text{Tr}(T^a)^4=16$  for any generator of the 16 spinor, and the local pure gauge anomaly, Eq. (3.4), has a nonvanishing irreducible part [26,33]. Therefore, a chiral six-dimensional  $SO(10)$  theory cannot be anomaly-free if it only contains copies of the 16 representation.

It was quite obvious from the beginning that the 3-generation, anomaly-free  $SU(5)$  solution Eq. (3.3) cannot be trivially promoted to an  $SO(10)$  model. As a 16 of  $SO(10)$  transform as  $\bar{5} \oplus 10 \oplus 1$  under  $SU(5)$ , the  $(\bar{5})_{L_6}$  and  $10_{R_6}$  of the third generation cannot originate from a single 16, which is either left- or right-handed. But since we have shown that a 16 of  $SO(10)$  is always anomalous in six dimensions, it cannot either yield a consistent  $SU(3) \otimes SU(2) \otimes U(1)$  theory.

To cancel this anomaly, one should either add another 16 with opposite chirality—but this amounts to losing the chirality of the theory—or add some other, new matter field, which spoils the simplicity researched when embedding  $SU(3) \otimes SU(2) \otimes U(1)$  or  $SU(5)$  in  $SO(10)$ . For example, one might consider two left-handed and a right-handed 16, plus a left-handed 10. The  $16_{R_6}$  and a  $16_{L_6}$  form a vector-like pair, and are therefore anomaly-free. The irreducible parts of the pure gauge anomalies of the remaining  $16_{L_6}$  and the  $10_{L_6}$  cancel [26,33], as will be obvious when we discuss  $E_6$ . The other local anomalies are in a sense harmless, since they can be canceled by the Green-Schwarz mechanism (reducible anomalies) or by adding gauge singlets (gravitational anomaly). Furthermore, there is no global gauge anomaly, because  $\Pi_6(SO(10))=0$ .

When  $SO(10)$  is broken into  $SU(5)$ , the 16 of this anomaly-free solution transform as in Eq. (3.1b), while the  $10_{L_6}$  yields  $(5 \oplus \bar{5})_{L_6}$ . All in all, we recover the fermion content of Eq. (3.3), with in addition a pair  $5_{L_6} \oplus (\bar{5})_{R_6}$ , in which the irreducible anomalies cancel, see Table I. This pair can be decoupled from the low energy spectrum by a Dirac mass term.

Let us now consider  $E_6$ . In that group, the basic, nonfactorizable traces are the traces over products of 2, 5, 6, 8, 9, and 12 generators [26]. Therefore, the local gauge anomaly is factorizable in  $D=6$ . There only remains a reducible part, which can be canceled through the Green-Schwarz mechanism, which also protects the theory against mixed anoma-

lies. Similarly, the gravitational anomalies of a single 27 can be canceled by adding singlet fermions with the appropriate chirality. Since the sixth homotopy group of  $E_6$  is trivial,  $E_6$  has no global gauge anomaly in six dimensions. Thus, a six-dimensional  $E_6$  theory, with a Green-Schwarz tensor and singlet fermions, is anomaly-free, whatever the number of 27s of the theory.

When the symmetry is broken, the absence of anomaly for a single 27 of  $E_6$  is transmitted to the  $SO(10)$  representation  $16 \oplus 10 \oplus 1$ . The singlet obviously has no gauge anomaly, and we recover the fact that the irreducible anomalies of a 16 and a 10 with identical chiralities cancel.

Going further in the sequence of Eq. (3.1a), the same idea suggests that the irreducible parts of the pure gauge anomalies of a 5, two  $\bar{5}$ , and a 10 of  $SU(5)$  with identical chiralities should cancel. They do indeed, as show the values of  $A_4(5,1)$ ,  $A_4(5,4)$ , and  $A_4(5,2)$  in Table I. This solution with three 5 or  $\bar{5}$  and a 10 is the minimal  $SU(5)$  fermion content free of irreducible anomalies in  $D=6$ . However, it does not contain all observed fermions of the standard model, which require at least 3 copies of this basic, anomaly-free building block. This then yields too many fermions, while there exists a more economical solution, Eq. (3.3). As mentioned before, the latter consists in fact of a left-handed copy of the minimal  $(\bar{5} \oplus \bar{5} \oplus \bar{5} \oplus 10)$  content, plus a vector-like, anomaly-free pair  $10_{L_6} \oplus 10_{R_6}$ .

### B. Constraints from anomalies in $D=8$

An important difference which characterizes the study of anomalies in eight dimensions is the absence of local gravitational anomalies, since the spacetime is  $4k$ -dimensional. This removes a condition which automatically led to the introduction of singlet fermions, which we shall now be able to discard, unless we wish to introduce some to cancel the global gravitational anomaly.

#### 1. $SU(3) \otimes SU(2) \otimes U(1)$ anomalies in $D=8$

When the spacetime dimension  $D \geq 8$ , theories based on the standard model group are less constrained than in lower dimensions. This is because almost all local gauge anomalies are now reducible: for  $SU(3)$  and  $SU(2)$ , both traces

$$STr(T^a T^b T^c T^d T^e) \quad (3.9)$$

and

$$STr(T^a T^b T^c T^d T^e T^f) \quad (3.10)$$

are factorizable, and therefore the corresponding anomalies can be canceled by Green-Schwarz tensors. So can be the  $[U(1)]^5$  and  $[U(1)]^6$  anomalies in most cases. Finally, gauge anomalies with bosons from different groups are always reducible.

Most mixed gauge-gravity anomalous diagrams are also reducible, with the exception of the  $D=8$  pentagonal diagram with four gravitons and a  $U(1)$  boson. This diagram

TABLE III. Coefficients in the symmetrized trace factorization for the lowest dimensions  $SU(5)$  representations in 8 dimensions.

$D=8$	$k$	$A_5(5,k)$	$A_5^{32}(5,k)$
5	1	1	0
10	2	-11	10
$\bar{10}$	3	11	-10
$\bar{5}$	4	-1	0

yields a contribution proportional to  $\text{Tr } Y$ . Thus, in  $D=8$ , the gauged fermions of a  $SU(3) \otimes SU(2) \otimes U(1)$  theory must satisfy the condition  $\text{Tr } Y = 0$ .

In eight dimensions, a simple solution consists in building a generation with left-handed versions of the  $\mathcal{Q}$ ,  $\mathcal{U}$ , and  $\mathcal{D}$  of Eq. (3.1c), and right-handed  $\mathcal{L}$  and  $\mathcal{E}$ . A single generation is then free of the only irreducible local anomaly, since  $\text{Tr } Y = 0$ . In that particular case,  $\text{Tr } Y^3 \neq 0$ , so that the  $[U(1)]^5$  anomaly is reducible [27].<sup>4</sup> Besides, such a generation contains  $N(3_{L_8}) - N(3_{R_8}) = 0$  triplets and two left-handed doublets. It therefore automatically satisfies the conditions, Eq. (2.5), necessary to cancel the  $SU(3)$  and  $SU(2)$  global anomalies, due to  $\Pi_8(SU(3)) = Z_{12}$  and  $\Pi_8(SU(2)) = Z_2$ . The only remaining constraint comes from the global gravitational anomaly, which requires either an even number of generations, or the introduction of an additional spin  $\frac{1}{2}$  singlet fermion.

Therefore, a single  $SU(3) \otimes SU(2) \otimes U(1)$  generation, Eq. (3.1c), with appropriate chirality choices, is anomaly-free, with the help of Green-Schwarz tensors to cancel the reducible anomalies and a Weyl fermion to protect the theory against the global gravitational anomaly. Anomalies give no restriction on the number of such generations.

#### 2. $SU(5)$ in eight dimensions

The anomalies of  $SU(5)$  in  $D=8$  are rather similar to the case  $D=6$ , since  $\Pi_9(SU(5)) = Z$  means that there is an irreducible local gauge anomaly, while  $\Pi_8(SU(5)) = 0$  guarantees the absence of global gauge anomaly.

The pure gauge anomaly Eq. (3.9) is non-factorizable, so that the irreducible part must be canceled by the fermion content. The expansion in terms of symmetrized traces over the generator of the basic representation involves the coefficients  $A_5$  given in Table III. Given the values of  $A_5(5,2)$  and  $A_5(5,4)$ , the most economic choice consists in taking  $n_g = 11$  generations ( $\bar{5} \oplus 10$ ), with appropriate chiralities: five  $((\bar{5})_{L_8} \oplus 10_{L_8})$  and six  $((\bar{5})_{L_8} \oplus 10_{R_8})$ .

The remaining, reducible part of the pure gauge anomaly, as well as the mixed anomaly, can be canceled through the Green-Schwarz mechanism. As announced above, there is no global gauge anomaly.

<sup>4</sup>On the other hand, in the case of a generation where all the fields of Eq. (3.1c) are left-handed, which satisfies  $\text{Tr } Y = 0$  as well, the chiralities are such that  $\text{Tr } Y^3$  also vanishes, and the pure gauge anomaly  $[U(1)]^5$  is not reducible.



Finally, the cancellation of the global gravitational anomaly necessitates an even number of spin  $\frac{1}{2}$  Weyl fermions. This requires the introduction of an odd number of singlet fermions. If one prefers not to add sterile matter, the global gravitational anomaly rules out the  $n_g=11$  solution which cancels the local gauge anomaly. The “minimal” anomaly-free solution then consists of twice the 11-generation solution, i.e., requires  $n_g=22$  reducible ( $\bar{5} \oplus 10$ ) generations.

When  $SU(5)$  breaks into  $SU(3) \otimes SU(2) \otimes U(1)$ , the 11-generation solution remains anomaly-free: the condition  $\text{Tr } Y$  is satisfied, since  $Y$  is a generator of  $SU(5)$ ; the eleven ( $\bar{5} \oplus 10$ ) yield  $N(3_{L_8}) - N(3_{R_8}) = -12$  triplets and  $N(2_{L_8}) - N(2_{R_8}) = 8$  doublets, so that there is no global gauge anomaly.

On the other hand, the  $D=8$  anomaly-free  $SU(3) \otimes SU(2) \otimes U(1)$  generation we found previously does not originate from  $SU(5)$  because of, for example, the opposite chiralities of the  $\mathcal{D}_{L_8}$  and  $\mathcal{L}_{R_8}$ . As in the case of six dimensions, an eight-dimensional  $SU(5)$  theory is more constrained than a theory based on  $SU(3) \otimes SU(2) \otimes U(1)$ .

### 3. $SO(10)$ and $E_6$ in eight dimensions

An  $SO(10)$  model suffers from the same problems in  $D=8$  as in six dimensions. Consider, once again, a single 16. The mixed gauge-gravitational anomaly involves traces over either one or three antisymmetric generators, and therefore vanishes. The local gauge anomaly Eq. (3.9), on the other hand, is nonvanishing for a 16: the trace over five  $SO(10)$  generators is proportional to the totally antisymmetric Levi-Civita tensor with 10 indices, see Appendix B. The possible remedies to cure this anomaly are the same as in six dimensions: either make the theory vector-like, or add matter fields in different  $SO(10)$  representations. In the latter case however, it becomes necessary to take care of the global gauge anomaly, arising from the homotopy group  $\Pi_8(SO(10)) = \mathbb{Z}_2$ , which will constrain the number of generations as in the case of  $SU(3) \otimes SU(2) \otimes U(1)$  in 6 dimensions. In addition, there must be an even number of spin  $\frac{1}{2}$  fermions, to cancel the global gravitational anomaly.

Using the same reasoning as in  $D=6$  dimensions, the eight-dimensional anomaly-free  $SU(5)$  theory with  $n_g=11$  generations we have encountered above does not come from an  $SO(10)$  theory, because of the  $(\bar{5})_{L_8} \oplus 10_{R_8}$  generations. There is therefore no contradiction with the impossibility of building anomaly-free  $SO(10)$  theories with the 16 representation.

To study the possible anomalies of a 27 in  $D=8$ , we shall not follow the same procedure as in the six-dimensional case. When  $E_6$  is broken into  $SO(10)$ , the 27 transforms following Eq. (3.1b). As always, no anomalies are created in this symmetry breaking. We have seen above that the 16 of  $SO(10)$  is anomalous. However, a 10 is anomaly-free, as can be checked by calculating the trace over the product of five real generators, Eq. (3.9). Therefore, a single 27 of  $E_6$  has an irreducible gauge anomaly in eight dimensions. Another, less

pedagogical way of seeing this anomaly consists in noticing that  $\Pi_9(E_6) = \mathbb{Z}$ .

The same reasoning allows us to check without calculation that a single 16 of  $SO(10)$  is anomalous in 6 or 8 dimensions. Since the  $\bar{5}$  and 10 have local anomalies which do not cancel (this is precisely why three or eleven generations are required), a parent 16 cannot be anomaly-free.

On the other hand,  $E_6$  has no mixed anomaly, since the 27 has no pure gauge anomaly in  $D=4$  dimensions. There is no global gauge anomaly either:  $\Pi_8(E_6) = 0$ . Finally, the global gravitational anomaly can as always be canceled through the introduction of an extra singlet.

Nonetheless, a chiral theory  $E_6$  with fermions only in the 27 representation is anomalous in  $D=8$  dimensions, due to the irreducible local gauge anomaly, which also spoils chiral eight-dimensional  $SO(10)$  theories.

## C. Anomalies in $D=10$

### 1. $SU(3) \otimes SU(2) \otimes U(1)$ in ten dimensions

As we have announced in Sec. III B, in  $D=10$ , there is in most cases no constraint on the possible  $SU(3) \otimes SU(2) \otimes U(1)$  gauged fermions arising from local anomalies: only the  $[U(1)]^6$  anomaly might be irreducible, under specific conditions. One may for example build a generation with left-handed versions of all fields. In that case,  $\text{Tr } Y^2 \neq 0$ , so that the  $[U(1)]^6$  anomaly is reducible [27]. Both  $SU(3)$  and  $SU(2)$  homotopy groups are non-trivial:  $\Pi_{10}(SU(3)) = \mathbb{Z}_{30}$  and  $\Pi_{10}(SU(2)) = \mathbb{Z}_{15}$ , and there might be global gauge anomalies. Since the number of triplets in a generation is even, a solution with  $n_g=15$  generations is obviously free of these anomalies. However, we shall see that information derived from  $SO(10)$  will allow us to improve this result: even a single generation will be found to have no global anomalies.

### 2. $SU(5)$ anomalies in $D=10$

Since  $SU(5)$  is a rank 4 group, there are four Casimir operators and as many basic traces, over products of 2, 3, 4, and 5 generators. Indeed, the trace of the product of six generators can be factorized in terms of these basic traces: in the case of the fundamental representation [14],

$$\begin{aligned} \text{STr}(t^a t^b t^c t^d t^e t^f) &= \frac{3}{4} \text{S}[\text{Tr}(t^a t^b t^c t^d) \text{Tr}(t^e t^f)] \\ &\quad - \frac{1}{8} \text{S}[\text{Tr}(t^a t^b) \text{Tr}(t^c t^d) \text{Tr}(t^e t^f)] \\ &\quad + \frac{1}{3} \text{S}[\text{Tr}(t^a t^b t^c) \text{Tr}(t^d t^e t^f)]. \end{aligned} \quad (3.11)$$

Therefore, after an expansion in traces over the generators of the fundamental representation, the pure gauge anomaly Eq. (3.10) is fully reducible, and so is the mixed anomaly. Accepting the introduction of the 2-, 4-, and 6-forms of the Green-Schwarz mechanism, these anomalies do not constrain the gauged fermion content, while it is impossible to cancel



all these anomalies with only an appropriate choice of matter content [14]. The gravitational anomaly does not vanish either, but can be canceled easily by singlet fermions. Therefore, nothing constrains the chiralities of the  $\bar{5}$  and 10 which constitute a generation, and we are free to choose them both left-handed, in order to recover the  $SU(3) \otimes SU(2) \otimes U(1)$  solution we proposed above.

The only stringent condition comes from the global gauge anomaly, due to  $\Pi_{10}(SU(5)) = \mathbb{Z}_{120}$ . Knowing the actual condition requires some knowledge of the coefficients  $c_{10}$  [see Eq. (2.5)] for the representations of  $SU(5)$  in 10 dimensions. In any case, models with  $120k$  left-handed generations are anomaly free, although more “economical” solutions exist, as we show hereafter.

### 3. $SO(10)$ and $E_6$ in ten dimensions

Let us now consider  $SO(10)$  in  $D=10$  dimensions. A single 16 representation suffers from irreducible gauge anomalies, since  $\Pi_{11}(SO(10)) = \mathbb{Z}$ , so that chiral theories containing only this representation are no more consistent than in  $D=6$  or  $D=8$ . Furthermore, for locally anomaly-free theories with an extended representation content, there is also a possible global anomaly due to  $\Pi_{10}(SO(10)) = \mathbb{Z}_4$ .

Consider for example a  $(16 \oplus 10 \oplus 10)$  representation. The irreducible part of the local gauge anomaly is zero, thanks to the choice of fermion content [26]. The remaining, reducible part of the anomaly and the mixed anomaly are canceled by Green-Schwarz tensors. One can easily cancel the gravitational anomaly with sterile fermions. Finally, four copies of  $(16 \oplus 10 \oplus 10)$  representations with identical chiralities will automatically have no global anomaly.

Let us break  $SO(10)$  into  $SU(3) \otimes SU(2) \otimes U(1)$ ; the resulting theory is necessarily anomaly-free. Each  $(16 \oplus 10 \oplus 10)_{L_{10}}$  yields eight left-handed triplets or antitriplets and as many doublets. Since the theory with broken symmetry has no anomaly, it is in particular free of global anomalies. Therefore, the conditions  $N(3_{L_{10}}) - N(3_{R_{10}}) = 32$  and  $N(2_{L_{10}}) - N(2_{R_{10}}) = 32$  are sufficient to cancel the  $SU(3)$  and  $SU(2)$  global anomalies. Inserting these results in Eq. (2.5), we now have further information on the coefficients  $c_{10}$  for the doublets and triplets: both  $c_{10}(2)$  and  $c_{10}(3)$  are multiple of 15. This allows us to significantly improve our condition for  $SU(3) \otimes SU(2) \otimes U(1)$  in ten dimensions: instead of 15 generations, we see that a single one is already anomaly-free.

Therefore, in  $D=10$ ,  $SU(3) \otimes SU(2) \otimes U(1)$  theories with any number of such generations are anomaly-free, and anomalies do not constrain  $n_g$ .

Since an  $SU(3) \otimes SU(2) \otimes U(1)$  generation of left-handed fermions is anomaly-free, so is an  $SU(5)$  generation: a single  $(\bar{5} \oplus 10)_{L_{10}}$  is already free of the potential global anomaly, while we only knew that a sufficient choice was  $n_g = 0 \bmod 120$ . This means that the coefficients  $c_{10}(\bar{5})$  and  $c_{10}(10)$  are both multiple of 120. As in the case of  $SU(2)$  and  $SU(3)$ , it is not necessary to actually calculate the coefficients to obtain interesting information.

We have just seen that a single  $(\bar{5} \oplus 10)_{L_{10}}$  is anomaly-free in  $SU(5)$ , while we mentioned previously that a 16 of  $SO(10)$  is anomalous. This means that the  $SU(5)$  theory cannot come from  $SO(10)$ , although the explanation is more subtle than in  $D=6$  or  $8$ . The symmetry breaking sequence Eq. (3.1a) could be further specified: in fact,  $SO(10)$  breaks into  $SU(5) \otimes U(1)$ , so that the  $\bar{5}$  and 10 of  $SU(5)$  also carry some  $U(1)$  charge (3 and  $-1$  respectively [34]), which gives rise to a new anomalous diagram  $[SU(5)]^5 U(1)$ . The contribution of this diagram, which is absent in a pure  $SU(5)$  theory, does not vanish, so that the  $SU(5) \otimes U(1)$  theory is not anomaly-free, whatever the number of generations, as the parent  $SO(10)$  theory.

The ten-dimensional pure gauge anomaly of the 27 of  $E_6$  can be checked in the same way as in  $D=8$ . Both 10 and 16 representations of  $SO(10)$  are anomalous, and the irreducible parts of their anomalies do not cancel [26]. Therefore, the 27 is necessarily anomalous, as also shows the homotopy group  $\Pi_{11}(E_6) = \mathbb{Z}$ . Although other anomalies either can be canceled (in the case of the mixed and gravitational anomalies) or vanish [ $\Pi_{10}(E_6) = 0$ , hence there is no global gauge anomaly], there is no way of obtaining an anomaly-free  $E_6$  theory with only copies of the 27, as in  $D=8$ .

In summary, in  $D=10$  it is possible to build chiral  $SU(3) \otimes SU(2) \otimes U(1)$  or  $SU(5)$  theories with any number of generations, but they do not come from  $SO(10)$  or  $E_6$  since a 16 or a 27 are always anomalous.

## IV. ANOMALY-FREE $SU(5)$ MODEL IN 6 DIMENSIONS

In this section, we investigate some of the properties of a six-dimensional model implementing the anomaly-free three-generation  $SU(5)$  fermion content of Eq. (3.3). A salient feature of these three generations is of course their non-identity. We shall first investigate whether this leads to specific characteristics at tree level (Sec. IV A). Then, we review in Sec. IV B some of the possible ways of going from  $SU(5)$  in  $D=6$  to the standard model in  $D=4$ .

### A. Basic properties of models with three nonidentical $SU(5)$ generations

As we have seen in Sec. III, the most stringent constraints imposed by anomalies on the fermion content affect six- and eight-dimensional  $SU(5)$  theories. The most promising case is obviously  $D=6$ , which hints at the necessary existence of three generations, as in the standard model.

A nice feature of the  $SU(5)$  fermion content in six dimensions is the difference between the third generation  $(\bar{5}_{L_6} \oplus 10_{R_6})$  and the other two. This might explain why the third standard model generation is so much heavier than the lightest two, and we shall assume that the  $\bar{5}_{L_6} \oplus 10_{R_6}$  yields, after breaking, the top, bottom,  $\tau$ , and  $\nu_\tau$ , although our discussion will not depend on this assumption.

Following this idea, let us examine the possible six-dimensional mass terms which could be given to the  $SU(5)$  fermions. The so-called minimal symmetry breaking scheme

for  $SU(5)$  in  $D=4$  involves two Higgs fields in the 5 and 24 representations. The former is used to break  $SU(5)$  and cannot give rise to a mass term. In fact, we shall see in Sec. IV B that this field is not even necessary to break the symmetry, if this is done through the compactification on an orbifold.

In opposition, the 5 Higgs field may yield  $SU(5)$  singlet terms, thanks to the tensor product decompositions [34]

$$\begin{aligned} 5 \otimes \bar{5} &= 1 \oplus 24 \\ \bar{5} \otimes \bar{5} &= \overline{10} \oplus \overline{15} \\ 10 \otimes 10 &= \bar{5} \oplus 45 \oplus 50 \\ \bar{5} \otimes 10 &= 5 \oplus \overline{45}. \end{aligned} \quad (4.1)$$

*A priori*,  $SU(5)$  invariance allows the construction of mass terms with two 10 or with a  $\bar{5}$  and a 10, independently of the spacetime dimension or the fermion chiralities. However, these two ingredients do influence mass terms, since the latter must be ( $D$ -dimensional) Lorentz invariant.

The chirality of the 10 of the third generation is opposite to that of all other  $\bar{5}$  and 10, so that a term involving this  $10_{R_6}$  and any other fermion can be Lorentz invariant. However, in order to have three light generations in 4D, such a term should be forbidden. [See the discussion following Eq. (3.3).] Furthermore, it is not possible to build a Lorentz invariant mass term with only left-handed fermions in  $D=6$ . Hence, at tree level, most, but not all, of the fermions will remain massless after symmetry breaking. Note that this feature is due to the different chirality assignment of the different generations in 6D, and not only to the structure of the gauge group. Of course, the situation will be modified by radiative corrections, which can generate mass terms after dimensional reduction to 4D. The rather restricted number of couplings will also affect the Cabibbo-Kobayashi-Maskawa (CKM) matrix and one might hope to obtain in a natural way some  $CP$  violation. In addition, compactification scheme leading to different localization patterns of the generations may further enrich the discussion. It would be interesting to see whether such a scheme, be it grand unified or not, could give rise to non-trivial mass textures. This would relate the hierarchy of fermion masses in the standard model to the cancellation of anomalies in 6D.

Before we turn to the issue of symmetry breaking and compactification, let us mention another prediction of the theory. In any grand unified theory (GUT), the weak mixing angle can be related to the traces over two generators of the group [35]:

$$\sin^2 \Theta_W = \frac{\text{Tr } T_3^2}{\text{Tr } Y^2}. \quad (4.2)$$

In four dimensions, if all the content of a single family is incorporated in one or more irreducible representations of the GUT group, the prediction does not depend on the group. These GUT representations could contain other states, but the latter have to be singlets under  $SU(3) \otimes SU(2) \otimes U(1)$ ,

and the result is thus the same for  $SU(5)$  and  $SO(10)$ , namely  $\sin^2 \Theta_W = 3/8$ . Nonetheless, radiative corrections have to be included, which strongly depend on the nature of the GUT group.

In our six-dimensional case, the predicted value for the first two families is of course the same as in four dimensions, since the chiralities are identical. In  $D=4$ , one could use charge conjugation to flip some of the chiralities thereby changing the corresponding signs, and the result would not change thanks to the square powers: this shows without calculation that the value is identical for the  $\bar{5}_{L_6} \oplus 10_{R_6}$ , despite the opposite chirality of the 10. Therefore, the value in our six-dimensional  $SU(5)$  is also  $\sin^2 \Theta_W = 3/8$ .

## B. Dimensional reduction and $SU(5)$ breaking

The combination of extra dimensions and GUT models brings together the issues related with both aspects. Thus, one should investigate the running of the standard model coupling constants, proton decay [35,36], the presence of monopoles, the hierarchy problem. The actual features of the various problems depend heavily on the localization and compactification mechanisms. Although we shall not go into detail in the present paper, we would like to make a few comments and suggestions to show that our approach is not grossly ruled out.

Starting from a  $SU(5)$  model in  $D=6$  dimensions, there are several paths towards  $SU(3) \otimes SU(2) \otimes U(1)$  in four dimensions. Two different steps are required, which may be simultaneous or not:  $SU(5)$  must be broken, and the six-dimensional theory must be reduced into a four-dimensional theory. The second operation should also involve some chirality selection, since the fermions of the standard model have definite, left-handed chiralities.

In the absence of a reliable mechanism to suppress the proton decay, a compactification scheme which allows  $SU(5)$  to survive in four dimensions should likely be discarded.

There has been recently a growth of interest in five-dimensional  $SU(5)$  models, either supersymmetric or not, with an orbifold  $S^1/Z_2$  or  $S^1/(Z_2 \otimes Z'_2)$  [33,37,36], as well as in  $D=6$   $SU(5)$  or  $SO(10)$  models with an orbifold  $T^2/(Z_2 \otimes Z'_2)$  [38,39]. In these studies, standard model fermions are totally confined to an orbifold fixed point, corresponding to our four-dimensional brane. The boundary conditions on the gauge bosons can be fixed appropriately so as to break  $SU(5)$  to  $SU(3) \otimes SU(2) \otimes U(1)$ : only the zero modes of the standard model gauge bosons have a non-zero value at the fixed point. This is an efficient way to reduce proton decay probability, since there is no overlap between the quarks and leptons and the  $SU(5)$ -specific, B-violating bosons.

In our six-dimensional case, if the standard model fields are strongly localized, for instance with a vortex [6,40], we may then benefit from the same effect. In addition, it has been showed that the localization of chiral fermions can enhance, through loop effects, the localization of the zero modes of the gauge bosons to which the fermions are coupled [41]. This further increases the suppression of un-

wanted  $SU(5)$  effects. Besides, possible standard model contributions to the proton decay may be suppressed by a residual spacetime symmetry, relic of the six-dimensional Lorentz invariance after compactification [42]. This mechanism is however model dependent. Whether a similar effect could suppress B-violating processes in the  $SU(5)$  model discussed here is an open question.

An issue related with the localization of the standard model fields is the size of the extra dimensions. If the fields are strongly localized, as seems to be necessary to avoid an important proton decay, then the extra dimensions might be large, with radii of the order of  $(10 \text{ TeV})^{-1}$ . This is also the grand unification energy scale, where the standard model couplings unify [43]. The unusual chirality of our third generation does not modify the actual value much.

To be fair, one must admit that, although the compactification on an orbifold has many advantages, since it allows both chirality selection and symmetry breaking, it is in no way predictive. First, it might lead to an anomalous four-dimensional theory if the 4D chiralities are not properly selected. Then, one could hope that the constraint of the absence of anomaly in four dimensions would only leave a single possibility, the standard model. Unfortunately, this is obviously not the case. As is clear from Eq. (3.3), with an appropriate choice of orbifold, we could as well get a single (anomaly-free) generation in  $D=4$ , plus vector-like fermions which could then be decoupled from the low energy spectrum. Therefore, even though anomalies might give an explanation of the existence of three generations, they are not restrictive enough so as to permit only one particular fermion content after dimensional reduction.

Another potentially interesting feature of our three-generation  $SU(5)$  solution is the necessary existence of three extra singlets which, after compactification, will give rise to Kaluza-Klein towers of sterile states [44]. In turn, these states can give masses to the light neutrinos.

Finally, the compactification does not wholly suppress the Green-Schwarz tensor which was required to cancel reducible  $SU(5)$  and mixed anomalies. There remain some of the tensor components, with axion-like couplings [45]. In an elaborate model, far beyond the scope of the present paper, these latter could be used to suppress the unobserved strong  $CP$  violation [46].

## V. CONCLUSIONS

As we have shown in this paper, anomaly cancellation restricts not only the chiral fermion content, but also the possible Yukawa couplings of grand unified theories propagating in extra dimensions.

Our results are summarized in Table IV. For instance, we have shown that  $SO(10)$  is not a good grand unification candidate, be it in  $D=6, 8$  or  $10$ , if the standard model fermions are represented by the 16. One needs to invoke other matter fields, beyond the standard model, in some other representation (see [39] for a similar discussion in the supersymmetric context). But this might mean that the relevant group is  $E_6$ , rather than  $SO(10)$ . On the other hand, six-dimensional  $SU(5)$  theories can be anomaly-free, provided

TABLE IV. Constraints on the gauged chiral fermion content of various theories in different spacetime dimensions.

	$D=6$	$D=8$	$D=10$
Standard model	$n_g \geq 2$	$n_g$ arbitrary	$n_g$ arbitrary
$SU(5)$	$n_g = 3k$	$n_g = 11k$	$n_g$ arbitrary
$SO(10)$	no solution with only copies of the 16		
$E_6$	$n_g$ arbitrary	no solution with 27 only	

the matter content is very finely tuned: if a generation consists of a representation  $\bar{5} \oplus 10$ , then the absence of anomalies necessitates a number of generations multiple of 3, with proper chiralities, see Eq. (3.3). However, note that anomalies do not impose that a generation be  $\bar{5} \oplus 10$ ; one could as well choose  $5 \oplus 10$ , or any such combination, although the condition remains  $n_g = 0 \pmod{3}$ . In fact, we have mentioned that the minimal anomaly-free fermion content for  $SU(5)$  in  $D=6$  is three 5 or  $\bar{5}$  with identical chirality, plus a 10 or  $\bar{10}$  with opposite chirality. This means that anomalies are not the ultimate answer; there must be another ingredient which must be combined with anomalies.

This additional ingredient might be some principle requiring that the theory be built with identical (including the chirality assignment) building blocks, as the standard model in four dimensions. In that case, the only anomalies which can give some restriction on the number of such building blocks are the global anomalies, both gauge and, to a lesser extent, gravitational. Stated differently, global anomalies are the only ones which can impose to have identical copies of a basic, necessarily locally anomaly-free, generation. Under this assumption, the only theories with anomaly-free fermion contents are  $SU(3) \otimes SU(2) \otimes U(1)$  in  $D=6$ , with  $n_g = 0 \pmod{3}$  [8], and six-dimensional  $E_6$ , eight- and ten-dimensional  $SU(3) \otimes SU(2) \otimes U(1)$ , and  $SU(5)$  in  $D=10$ , all of which are anomaly-free whatever the number of identical generations.

Our six-dimensional  $SU(5)$  solution, where the number of generations is imposed by the absence of the local gauge anomaly, does not satisfy this criterion. Nonetheless, the nonidentity of the generations might be a blessing and could give rise to interesting phenomenological predictions, for instance regarding the hierarchy of fermion masses in the standard model.

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## APPENDIX A: CHIRALITY AND C IN EVEN DIMENSIONS

To fix the notations we use throughout the paper, and for completeness’ sake, we recall in this Appendix some basic definitions and properties about chiral fermions in Euclidean and Minkowski space [47].

### 1. Chirality and even dimensions

The Lorentz group will be denoted  $SO(t, s)$  where  $t$  and  $s$  are the numbers of time and space dimensions respectively ( $D = t + s$ ). A Dirac spinor obeys the following transformation law under infinitesimal Lorentz transformations:

$$\delta\Psi \equiv -\frac{1}{2}\omega_{MN}\Sigma_{(D)}^{MN}\Psi, \quad M, N = 0, \dots, D-1, \quad (\text{A1})$$

where  $\omega_{MN}$  are real coefficients and  $\Sigma_{(D)}^{MN}$  denote the Lorentz generators in spinorial representation. The latter are given in terms of Dirac  $2^{[D/2]} \times 2^{[D/2]}$  matrices  $\Gamma_{(D)}$ , where  $[A]$  is the integer part of  $A$ , which satisfy the Clifford algebra:

$$\Sigma_{(D)}^{MN} \equiv \frac{1}{4}[\Gamma_{(D)}^M, \Gamma_{(D)}^N], \quad (\text{A2})$$

$$\{\Gamma_{(D)}^M, \Gamma_{(D)}^N\} = 2\eta_{MN}. \quad (\text{A3})$$

The matrix  $\eta_{MN}$  is the flat metric with signature  $(t, s)$ . In the following, we shall drop the subscript  $(D)$ , except when there may be some ambiguity.

In any even dimension  $D$ , one can introduce the matrix  $\bar{\Gamma}$ , which is the analog of  $\gamma_5$  in  $D=4$ :

$$\bar{\Gamma} \equiv \alpha \Gamma^0 \Gamma^1 \dots \Gamma^{D-1} \quad (\text{A4})$$

where the coefficient  $\alpha$  is conventionally chosen such as  $(\bar{\Gamma})^2 = 1$ . This matrix anticommutes with all Dirac matrices, and thus commutes with all Lorentz generators  $\Sigma^{MN}$ . The latter property means that the Dirac representation is reducible: a Dirac spinor  $\Psi$  splits in two irreducible parts  $\Psi_+$  and  $\Psi_-$ , called Weyl spinors, which transform independently under Lorentz transformations. The Weyl spinors are defined as follows:

$$\Psi_{\pm} \equiv \frac{1}{2}(1 \pm \bar{\Gamma})\Psi. \quad (\text{A5})$$

Throughout the paper, we replace  $\Psi_+$  and  $\Psi_-$  with  $\Psi_{R_D}$  and  $\Psi_{L_D}$  respectively.

In odd dimension  $D$ , the set of Dirac matrices consists of the  $D-1$  Dirac matrices in dimension  $D-1$ , plus an additional one which is proportional to  $\bar{\Gamma}_{(D-1)}$ . Thus, it is no longer possible to have another matrix which could anticommute with all matrices  $\Gamma_{(D)}$ . This prevents the definition of chirality (and therefore chiral fermions or anomalies) in odd dimensions.

### 2. Charge conjugation in $4k$ and $4k+2$ dimensions

Since there is only one faithful representation of the Clifford algebra with a given dimension, all sets of matrices which satisfy Eq. (A3) are related by similarity transformations. As the set of the complex conjugate matrices  $(\Gamma^M)^*$  fulfils this requirement, there exists a matrix  $B$  such that

$$(\Gamma^M)^* = B\Gamma^M B^{-1}, \quad (\Sigma^{MN})^* = B\Sigma^{MN} B^{-1}. \quad (\text{A6})$$

Using this relation, one can define a charge conjugate Dirac spinor  $\Psi^c$  which transforms exactly in the same way as  $\Psi$  [Eq. (A1)]:

$$\Psi^c \equiv \mathbf{C}\Psi \equiv B^{-1}\Psi^* \quad (\text{A7})$$

$$\delta(\Psi^c) \equiv -\frac{1}{2}\omega_{MN}\Sigma^{MN}\Psi^c \quad (\text{A8})$$

where  $\mathbf{C}$  is the charge conjugation operator.

The transformation of a chiral, Weyl fermion under charge conjugation depends on the numbers of space and time dimensions. When  $(s-t)/2$  is odd,  $\{\mathbf{C}, \bar{\Gamma}\} = 0$ , so that charge conjugation flips chirality; otherwise,  $\mathbf{C}$  and  $\bar{\Gamma}$  commute, and charge conjugation does not modify the chirality of a Weyl fermion. In the “usual” case with a single time dimension  $t=1$ , which we assume from now on, this gives

$$\begin{aligned} \{\mathbf{C}, \bar{\Gamma}\} &= 0 \quad \text{in} \quad D = 4k, \\ [\mathbf{C}, \bar{\Gamma}] &= 0 \quad \text{in} \quad D = 4k+2. \end{aligned} \quad (\text{A9})$$

$\mathbf{C}$  flips chirality in  $D=4k$ , while it does not in  $D=4k+2$ .

### 3. Chiral representation

We recall a possible explicit realization of the Dirac  $\Gamma$  matrices for a spacetime dimension  $D$ , which has the attractive feature to separate left- and right-handed fermions. This representation will also be useful for the trace calculation in Appendix B.

In dimension  $D$ , the matrices are built from the Dirac matrices in dimension  $D-1$  following:

$$\begin{aligned} \Gamma_{(D)}^0 &= \begin{bmatrix} 0 & \mathbf{1}_{D/2} \\ \mathbf{1}_{D/2} & 0 \end{bmatrix}, \\ \Gamma_{(D)}^k &= \begin{bmatrix} 0 & \Gamma_{(D-1)}^0 \Gamma_{(D-1)}^k \\ -\Gamma_{(D-1)}^0 \Gamma_{(D-1)}^k & 0 \end{bmatrix}, \\ \Gamma_{(D)}^{D-1} &= \begin{bmatrix} 0 & \Gamma_{(D-1)}^0 \\ -\Gamma_{(D-1)}^0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\bar{\Gamma}_{(D)} = \begin{bmatrix} \mathbf{1}_{D/2} & 0 \\ 0 & -\mathbf{1}_{D/2} \end{bmatrix}, \quad (\text{A10})$$

where  $k=1, \dots, D-2$  and  $\mathbf{1}_{D/2}$  is the  $D/2$ -dimensional unity matrix. As recalled above, the matrices  $\Gamma_{(D-1)}^k$  where  $k$  runs from 1 to  $D-3$  are the Dirac matrices  $\Gamma_{(D-2)}^k$  in dimension  $D-2$ , and  $\Gamma_{(D-1)}^{D-2} = i\Gamma_{D-2}$ . Since the Lorentz generators  $\Sigma_{(D)}^{MN}$ , Eq. (A2), are commutators of Dirac matrices, they are all block diagonal, made of two  $D/2 \times D/2$  blocks. These blocks yield the generators of the transformation for the  $\Psi_{L_D}$  and  $\Psi_{R_D}$ .



APPENDIX B:  $SO(10)$  TRACES IN 8 DIMENSIONS

In this appendix, we show that the symmetrized trace over the product of five  $SO(10)$  generators does not vanish in general, and more precisely is proportional to the Levi-Civita tensor with 10 indices. This is analogous to the fact that in  $D=4$  dimensions, the triangle anomaly vanishes for every group  $SO(N)$  except if  $N=6$ , the trace over three  $SO(6)$  generators being proportional to the six-indices Levi-Civita tensor.

We are interested in symmetrized traces which involve 5 generators  $T$  for the basic 10 and the spinorial 16 representations of  $SO(10)$ , which yield the local gauge anomalies of the corresponding representations in  $D=8$ . These symmetrized traces, Eq. (3.9), can be written

$$\text{STr}((T^a)^{\alpha\beta}(T^b)^{\gamma\delta}(T^c)^{\kappa\lambda}(T^d)^{\mu\nu}(T^e)^{\rho\sigma}), \quad (\text{B1})$$

where all Greek indices run from 0 to 9. The symmetrized trace is invariant under orthogonal  $SO(10)$  transformations of the generators. In terms of Minkowski instead of Euclidean spacetime, the trace Eq. (B1) is a  $SO(1,9)$  invariant, which means that it is proportional to a 10-indices tensor of a  $D=10$  theory with appropriate symmetry properties. The only possible one is the totally antisymmetric Levi-Civita tensor  $\epsilon_{\alpha\beta\gamma\delta\kappa\lambda\mu\nu\rho\sigma}$ . We only have to check whether the proportionality constant is zero or not, for each representation we consider.

The representation 10 is real [by definition of  $SO(10)$ ], and thus all generators are similar to their complex conju-

gate:  $STS^{-1} = -T^*$ . Since the trace involves the product of 5 generators, it is equal to the opposite of the trace over the conjugate generators. On the other hand, the generators are hermitian, so that both traces over either the generators or their complex conjugates are equal. Therefore, the trace Eq. (B1) vanishes for the 10.

The (Dirac) spinor, reducible 32 representation of  $SO(10)$  is also a representation of  $SO(1,9)$ , generated by the  $\Sigma_{(10)}^{MN}$ , and we may use the choice of generators of Sec. A 3, in the case  $D=10$ . These are antisymmetric,  $32 \times 32$  matrices, composed of two  $16 \times 16$  blocks, which transform  $D=10$  left- and right-handed Weyl fermions, which are precisely the 16 and  $\bar{16}$  of  $SO(10)$ . As noted in previous section, these blocks involve the  $D=8$  Dirac matrices  $\Gamma_{(8)}^M$ , plus  $\bar{\Gamma}_{(8)}$ . Let us take the upper block to describe our 16 representation. Calculating the trace of a product of 16 is now straightforward: we just compute the product of the  $32 \times 32$  matrices  $\Sigma_{(10)}^{MN}$  and then evaluate the trace over the upper block of the product.

Consider the product  $\Sigma_{(10)}^{01}\Sigma_{(10)}^{23}\Sigma_{(10)}^{45}\Sigma_{(10)}^{67}\Sigma_{(10)}^{89}$ . The upper block is proportional to

$$\Gamma_{(8)}^0\Gamma_{(8)}^1\cdots\Gamma_{(8)}^7\bar{\Gamma}_{(8)}\propto(\bar{\Gamma}_{(8)})^2, \quad (\text{B2})$$

which is the identity matrix  $\mathbf{1}_{16}$ . Thus, the trace is non-zero, and so is the proportionality coefficient with the Levi-Civita tensor.

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